MA206X AY26-2 — Block I Practice Review

Probability & Random Variables (Lessons 6–14)

This review is for practice only and is not graded.

Instructions: Work each problem completely, starting with the appropriate probability statement or random variable definition where indicated. Answers may be left in exact form unless otherwise stated. You are authorized your SRC and calculator.

Concept Check

Answer each of the following. These are quick checks of key definitions and properties.

1. True or False: If $P(A) = 0.5$ and $P(B) = 0.3$, then A and B must be mutually exclusive.
2. True or False: The CDF of any random variable is a non-decreasing function.
3. Fill in the blank: For independent events A and B, $P(A \cap B) =$ ________________.
4. True or False: If $X \sim \text{Bin}(n, p)$, then $E(X) = np$ and $\text{Var}(X) = np(1-p)$.
5. Fill in the blank: If events A and B are mutually exclusive, then $P(A \cup B) =$ ________________.
6. True or False: A valid PMF must satisfy $\sum_{\text{all } x} P(X = x) = 1$ and $P(X = x) \geq 0$ for all x.
7. Fill in the blank: For a continuous random variable with pdf $f(x)$, $P(a \leq X \leq b) =$ ________________.
8. True or False: If X is a continuous random variable, then $P(X = 3) = 0$.
9. Fill in the blank: The variance of a random variable can be computed as $\text{Var}(X) =$ ________________.
10. True or False: The exponential distribution has the memoryless property, meaning $P(T > s + t \mid T > s) = P(T > t)$.

Probability Basics (Lessons 6–8)

Problem 1. A survey of 500 college students found that 280 play video games (V), 200 play a musical instrument (I), and 90 do both.

(a) Find $P(V \cup I)$.

(b) Find $P(V^c \cap I^c)$, the probability a student does neither.

(c) Are V and I independent? Justify with a calculation.

Problem 2. A fair six-sided die is rolled twice. Let A be the event the first roll is even and B be the event the sum of both rolls is at least 9.

(a) List the sample space outcomes in B.

(b) Find $P(B)$.

(c) Find $P(A \cap B)$.

(d) Are A and B independent?

Problem 3. Suppose $P(A) = 0.3$, $P(B) = 0.5$, and A and B are independent.

(a) Find $P(A \cap B)$.

(b) Find $P(A \cup B)$.

(c) Find $P(A^c \cap B^c)$.

(d) Find $P(B|A^c)$.

Problem 4. A medical test for a rare disease has the following characteristics: 2% of the population has the disease. The test correctly identifies 95% of those who have the disease (sensitivity). The test correctly identifies 90% of those who do not have the disease (specificity). A randomly selected person tests positive.

(a) Define appropriate events and list all given probabilities.

(b) Use the Law of Total Probability to find $P(T)$, the probability of a positive test.

(c) Use Bayes' Rule to find the probability the person actually has the disease given a positive test.

(d) Interpret your answer to (c) in context.

Problem 5. A restaurant offers 3 appetizers, 5 entrées, and 4 desserts.

(a) How many different three-course meals (one appetizer, one entrée, one dessert) are possible?

(b) If you invite 3 friends and each independently picks an entrée at random, what is the probability all three choose different entrées?

Discrete Random Variables & PMFs (Lesson 9)

Problem 6. A carnival game costs \$2 to play. You draw one card from a standard 52-card deck. If you draw an ace, you win \$10. If you draw a face card (J, Q, K), you win \$3. Otherwise, you win nothing. Let X be your **net** profit.

(a) Define the PMF of X in a table.

(b) Find $E(X)$. Is this a fair game?

(c) Find $\text{Var}(X)$.

Problem 7. Let X be a discrete random variable with the following PMF:

$$P(X = x) = \begin{cases} c \cdot x^2 & \text{for } x = 1, 2, 3 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find the value of c that makes this a valid PMF.

(b) Find the CDF $F(x)$ and write it in piecewise form.

(c) Find $P(X \geq 2)$.

(d) Find $E(X)$ and $\text{Var}(X)$.

Binomial Distribution (Lesson 10)

Problem 8. A multiple-choice quiz has 15 questions, each with 4 options. A student guesses randomly on every question. Let X be the number of correct answers.

(a) Define the random variable and justify why a binomial model is appropriate.

(b) What is the expected number of correct answers?

(c) What is the standard deviation of X?

(d) Find $P(X = 5)$.

(e) Find $P(X \geq 2)$.

Problem 9. A quality control inspector examines a batch of 20 light bulbs, each independently having a 10% defect rate.

(a) What is the probability that exactly 2 are defective?

(b) What is the probability that none are defective?

(c) What is the variance?

Poisson Distribution (Lesson 11)

Problem 10. A bookstore receives an average of 5 online orders per hour.

(a) Define the appropriate random variable.

(b) Find the probability of receiving exactly 3 orders in the next hour.

(c) Find the probability of receiving no orders in the next hour.

(d) Find the probability of receiving at least 2 orders in the next hour.

(e) What is the probability of receiving exactly 8 orders in a two-hour period?

Problem 11. A city averages 3 power outages per month. Assume outages follow a Poisson process.

(a) What is the probability of exactly 5 outages in a given month?

(b) What is the probability of no outages in a two-week period? (Assume 1 month $\approx$ 4 weeks.)

Continuous Random Variables & PDFs (Lesson 12)

Problem 12. The continuous random variable X has pdf:

$$f(x) = \begin{cases} kx(1-x) & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find k.

(b) Find the CDF $F(x)$. Write it in full piecewise form.

(c) Find $P(0.25 \leq X \leq 0.75)$.

(d) Find $E(X)$.

(e) Find $\text{Var}(X)$.

Problem 13. The continuous random variable Y has pdf:

$$f(y) = \begin{cases} \frac{3}{16}\sqrt{y} & 0 \leq y \leq 4 \\ 0 & \text{otherwise} \end{cases}$$

(a) Verify this is a valid pdf.

(b) Find $P(Y > 1)$.

(c) Find $E(Y)$.

Normal Distribution (Lesson 13)

Problem 14. The time (in minutes) it takes to complete an online checkout is normally distributed with $\mu = 8$ minutes and $\sigma = 2$ minutes. Let X denote checkout time.

(a) Find $P(X > 12)$.

(b) Find $P(5 < X < 11)$.

(c) Find the checkout time that is exceeded by only 5% of customers.

(d) What proportion of checkout times are within 1.5 standard deviations of the mean?

Problem 15. SAT math scores are approximately $N(520, 100^2)$.

(a) What score corresponds to the 90th percentile?

(b) What proportion of students score between 400 and 650?

(c) If a scholarship requires a score in the top 2%, what is the minimum qualifying score?

Exponential Distribution (Lesson 14)

Problem 16. Customers arrive at a coffee shop at an average rate of 10 per hour. Let T be the waiting time (in hours) between consecutive customers.

(a) Define the random variable using proper notation. What are $E(T)$ and $\text{Var}(T)$?

(b) Find the probability that the wait between two consecutive customers exceeds 15 minutes (0.25 hours).

(c) Find the probability that the next customer arrives within 6 minutes (0.1 hours).

(d) Find the median wait time.

Problem 17. Lightning strikes occur in a national park at a rate of 6 per week during summer. Let T be the time (in weeks) between successive lightning strikes.

(a) What distribution does T follow? State the parameter.

(b) Find $P(T > 0.5)$, the probability of waiting more than half a week.

(c) Use the memoryless property: Given that no strike has occurred for 2 days (2/7 weeks), what is the probability of no strike for 3 more days (3/7 weeks)?

Mixed Practice

Problem 18. For each scenario, identify the most appropriate probability distribution (Binomial, Poisson, Exponential, Normal) and define the random variable with its parameter(s). You do **not** need to compute a probability.

(a) The number of typos on a randomly selected page of a 300-page novel, if there are on average 1.5 typos per page.

(b) The number of heads in 50 flips of a fair coin.

(c) The time until the next customer service call, if calls arrive at 8 per hour.

(d) The weight of a randomly selected bag of flour from a production line where bags average 5.0 lb with standard deviation 0.1 lb.

(e) The number of commercial flights out of 400 that experience a delay, if each flight independently has a 15% chance of delay.

Problem 19. A factory has two machines, A and B. Machine A produces 60% of all items and Machine B produces 40%. The defect rate for Machine A is 3% and for Machine B is 5%. An item is selected at random.

(a) What is the probability the item is defective?

(b) Given the item is defective, what is the probability it came from Machine B?

Problem 20. The continuous random variable X has CDF:

$$F(x) = \begin{cases} 0 & x < 0 \\ \frac{x^2}{9} & 0 \le x \le 3 \\ 1 & x > 3 \end{cases}$$

(a) Find the pdf $f(x)$.

(b) Find $P(1 \le X \le 2)$.

(c) Find the median of X.

(d) Find $E(X)$.